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## TRANSLATION

VAPORIZATION OF SOLIDS BY ABSORBED RADIATION ENERGY

By

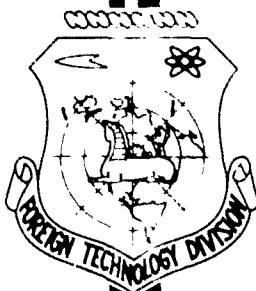
E. M. Shakhov

## FOREIGN TECHNOLOGY DIVISION

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# UNEDITED ROUGH DRAFT TRANSLATION

VAPORIZATION OF SOLIDS BY ABSORBED RADIATION ENERGY

BY: E. M. Shakhov

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# VAPORIZATION OF SOLIDS BY ABSORBED RADIATION ENERGY

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The heating up of a solid body absorbing radiant energy was discussed in earlier communications (Refs. 1, 2). When the density of the radiation flux is sufficiently high, the body surface may acquire a very high temperature at which the substance will inevitably undergo some physicochemical transformations (melting, evaporation, decomposition, etc.). Let us consider the case where the body absorbs a high-power radiation pulse  $Q$  of short duration  $\tau$ . By the end of the time interval  $\tau$  the surface temperature will have reached a value proportional to  $Q/\sqrt{\tau}$ ; i.e., when  $Q$  is sufficiently large and  $\tau$  is sufficiently small, the temperature may rise to any conceivable level. Therefore, to evaluate the heating effect of powerful radiation fluxes absorbed by a solid body, it is essential to consider the attendant physicochemical transformations of the substance. Of all such phenomena, melting is the simplest and most thoroughly explored (Refs. 3, 4). The present study is concerned with the unidimensional problem which involves the vaporization of a solid body heated by the incident radiant energy it absorbs. The absorption is assumed to occur on the

surface, and the radiation flux is taken to be the equivalent of a heat flow  $q$  along the boundary whose magnitude depends on surface temperature and time  $t$ . Two cases of evaporation are considered. In the first case the body, initially, is contiguous to a vacuum; in the second, the ambient space, at some distance from the body, is filled with gas. Both vapor and gas are treated as perfect gases. The mechanism of vaporization is however substantially different in the two cases (Refs. 5, 6). When evaporation is intense, in the case of a body initially contiguous to a vacuum, the rate at which the vaporization products are removed is determined by the velocity of expansion. The latter is a gas-dynamic process, which is sufficiently rapid and cannot limit the speed of the entire process. For this reason the loss of solid mass  $m$  per unit of evaporating surface proceeds at about the same rate as evaporation into a vacuum  $m_v$ . In the second case, however, where the ambient space is filled with a gas-vapor mixture, the evaporation products are removed from the surface by diffusion, at a rate which is low as compared with the rate of evaporation into a vacuum - the highest possible for a given temperature of the body surface. What happens, apparently, is that most molecules which have left the body and entered the gas phase return to the surface and once again become densely packed (condensation), while only a few are permanently removed. It follows that a near-equilibrium is maintained in the process, and the state of vapor enveloping the surface is close to saturation.

# I. Evaporation from the surface of a body initially contiguous to a vacuum

1. We shall consider the relatively simple case of a solid -- a half-space  $x < 0$  -- whose initial temperature is uniform throughout. The body absorbs radiant heat along the boundary with an intensity  $q$  dependent on surface temperature and time. Since the body occupies a negative region of space, the heat flux  $q$  should also be regarded as negative. Let  $\theta$  denote the difference between the initial body temperature and instantaneous temperature at any arbitrary point  $x$  of the body. It is assumed that heat propagation within this solid obeys the linear equation of thermal conduction

$$\frac{\partial \theta}{\partial t} = \kappa \frac{\partial^2 \theta}{\partial x^2}, \quad (1.1)$$

and that vapor motion is described by equations of gas dynamics. On evaporation the body surface changes its position in space. Let us designate the law governing its motion by  $x_0(t)$ ; the translation velocity, by  $\dot{x}_0(t)$ ; and by  $\rho_0$ ,  $p_0$ ,  $T_0$  and  $u_0$ , the density, pressure, absolute temperature and mass velocity of a vapor particle in the neighborhood of the evaporating surface. From the laws of mass and energy conservation the following expressions can now be derived interrelating the unknown parameters of the evaporating surface:

$$-\rho \dot{x}_0(t) = \rho_0(u_0 - \dot{x}_0(t)) = \dot{m}, \quad (1.2)$$

$$L\dot{m} = -q - \lambda \frac{\partial \theta}{\partial x}. \quad (1.3)$$

Here  $\rho$ ,  $\lambda$  are the density and coefficient of heat conductivity of a solid body, and  $L$  is the specific heat of vaporization taken to

be a known function of temperature.

What we know of conditions expressed by (1.2), (1.3), in conjunction with initial data and boundary conditions at infinity, is not enough to determine a unique solution of eq. (1.1) or a system of gas-dynamic equations, because our knowledge of boundary conditions existing on the evaporating surface  $x = x_0(t)$  is inadequate (Ref. 7). From the standpoint of physics, this is quite obvious, since conditions (1.2) and (1.3) do not reflect the specificity of the process. For additional relationships governing an evaporating surface we must turn to thermodynamics, or to the molecular-kinetic theory of the structure of matter.

We shall attempt to determine in what way the mass evaporation rate depends on the temperature of body surface and the state of vapor in the neighborhood of this surface. The rate of any physico-chemical process can be expressed with sufficient accuracy as the difference between the respective rates of the direct and reverse processes (Ref. 8). For the case of evaporation, we have

$$\dot{m} = \dot{m}_+ - \dot{m}_- \quad (1.4)$$

where  $\dot{m}_+$  is the direct process rate, or the rate of evaporation into a vacuum, which depends on surface temperature alone;  $\dot{m}_-$  is the rate of a reverse process, i.e., condensation, dependent on the state of vapor near the body surface. The values of  $\dot{m}_+$  and  $\dot{m}_-$  can be calculated on the basis of statistical physics. For the rate of mass evaporation into a vacuum we have (Ref. 9)

$$\dot{m}_+ = (1 - r) \frac{p_s}{\sqrt{2\pi RT_s}} \quad (1.5)$$



where  $T_s$  is the absolute temperature of body surface;  $p_s$  is the saturated vapor pressure corresponding to this temperature;  $R$  is the gas constant of vapor;  $r$  is reflection factor, i.e., the fraction of the total number of molecules that reaches the body surface and undergoes condensation. To calculate  $\dot{m}_-$  we shall assume, as did earlier investigators (Ref. 10), that an equilibrium distribution of molecular velocities exists near the body surface, the average velocity being equal to  $u_0$ :

$$\dot{m}_- = -(1-r) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dv_x dv_y \int_{-\infty}^{\infty} f v_x dv_z$$

Here,

$$f = \frac{1}{(2\pi RT_s)^{3/2}} \exp \left\{ -\frac{(v_x - u_0)^2 + v_y^2 + v_z^2}{2RT_s} \right\}$$

is the Maxwellian function of velocity distribution. Evaluation of the integral yields

$$\dot{m}_- = (1-r) \frac{p_s}{\sqrt{2\pi RT_s}} \operatorname{ierfc} \bar{u}_0$$

where  $\bar{u}_0 = \frac{u_0}{\sqrt{2RT_s}} = M_0 \sqrt{\frac{\gamma}{2}}$ ;  $M_0$  is the Mach number near the body surface;  $\gamma$  is the ratio of the specific heats of the vapor;

$$\operatorname{ierfc} x = \frac{1}{\sqrt{\pi}} e^{-x^2} + x \frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-t^2} dt$$

For the mass evaporation rate we now obtain the expression

$$\dot{m} = \dot{m}_+ - (1-r) \frac{p_s}{\sqrt{2\pi RT_s}} \operatorname{ierfc} \bar{u}_0 \quad (1.6)$$

From condition (1.2) it follows that  $\frac{u_0}{\sqrt{2RT_s}} \ll 1$ , since  $\frac{p_s}{p} \ll 1$ ; we can therefore write, with high accuracy

$$\dot{m} = p_s \mu_0 \quad (1.2')$$

Comparison of (1.2') with (1.6) gives

$$\dot{m} = \frac{2\bar{u}_0}{2\bar{u}_0 + i\epsilon r c \bar{u}_0} \dot{m}_+ \quad (1.7)$$

Consequently, the ratio for the loss of solid mass in the two cases - evaporation into an ambient space filled with pure vapor as against evaporation into a vacuum - will depend solely on the Mach number value near the evaporating surface. Function  $\dot{m}/\dot{m}_+$  rises monotonically, and very rapidly, with the rise of the Mach number. For a monoatomic gas ( $\gamma = 5/3$ ), when  $M_0 = 0.5$ ,  $\dot{m}/\dot{m}_+ = 0.805$  while when  $M_0 = 1$ ,  $\dot{m}/\dot{m}_+ = 0.964$ .

It will next be shown that the Mach number cannot be much less than unity. In fact, for  $M_0 \ll 1$  we have from (1.4) and (1.7), with an accuracy limited by terms containing  $M_0$  in the zero power --  $\dot{m}_+ = \dot{m}_-$ ; i.e., the state of vapor near body surface is close to saturation. Let the parameters of saturated vapor be denoted by the subscript \*. Evaluating the ratio of the velocity at which the evaporation front is propagated to the speed of sound in saturated vapor will give

$$\frac{\dot{r}_*(t)}{c_*} = \frac{\dot{m}}{\rho_* c_*} \leq \frac{\dot{m}_+}{\rho_* c_*} \leq \frac{1}{\sqrt{2\pi}} \frac{P_*}{P} \ll 1.$$

This means that when the vapor is in a nearly saturated state, the front of an expansion wave is propagated at a speed which substantially exceeds the velocity of a shifting evaporation front. The rarefaction waves will reach the evaporating surface, lowering the speed of sound and simultaneously reducing the gas density near the surface. This in turn will reduce the condensation rate  $\dot{m}_-$ , hence both mass velocity and the Mach number will increase. It follows that should the vapor reach a state close to saturation

(as in the case where the vapor region is initially limited), this condition will soon be upset. If, on the other hand, the vapor region is consistently bound by a vacuum, the vapor will never reach a state close to saturation.

Qualitative considerations, along the same lines as those set forth above, suggest that in general, near the body surface, the velocity at which the front of an expansion wave propagates cannot materially exceed the velocity of a shifting evaporation front. Now, since

$$\frac{u_0 - a_0}{x_0(t)} = \frac{p_0}{p_0} (1 - M_0),$$

it follows that for  $M_0 < 1$

$$1 - M_0 \frac{p_0}{p_0} \frac{u_0 - a_0}{x_0(t)} \ll 1.$$

The Mach number is therefore either greater than unity or (for  $M_0 < 1$ ) is close to unity. Consequently, on the basis of (1.7) we can approximate  $\bar{r} = \bar{r}_+$ . This assumption seems all the more justified since the reflection factor  $r$ , which enters into  $\bar{r}_+$ , is determined experimentally, with an accuracy not exceeding 5% (Ref. 11).

In summing up, we conclude that the thermal problem on the heating up of a body able to evaporate can be solved independently of determining the gas motion, and is formulated as follows. It is required to find a temperature field within the body, by way of solving the heat conductivity equation (1.1), under conditions existing along an unknown boundary  $x_0(t)$

$$m = m_+ \quad \psi(0, t) \quad (1.8)$$

$$\frac{dm}{dt} = \frac{L}{\lambda} \frac{\partial \psi}{\partial x} \bigg|_{x=0} = \frac{L}{\lambda} m \quad \psi(t, 0) \quad (1.9)$$

and under the initial condition

$$\theta(x, 0) = 0. \quad (1.10)$$

Here  $\theta_*$  is the value of function  $\theta_*(x, t)$  on the surface. A distinctive feature of this nonlinear problem stems from the physical fact that a substance is capable of evaporating under any conditions but at a different rate, which depends largely on the surface temperature. The peculiarity here is that this surface "temperature"  $\theta_*(t)$  is not preassigned but must be determined, along with the entire temperature field and the unknown mobile boundary.

2. We shall next show that a unique solution is actually possible for the problem as formulated above. To this end a solution of (1.1) (Ref. 12) in the form of an integral is given. Under conditions expressed in (1.8)-(1.10), it is written in the form

$$\begin{aligned} \theta(x, t) = & x \int_0^t G(x, t; x_0(\tau), \tau) \psi(\tau; \theta_*) d\tau + \\ & + \int_0^t \left[ G(x, t; x_0(\tau), \tau) \dot{x}_0(\tau) - x \frac{\partial G}{\partial x}(x, t; x_0(\tau), \tau) \right] \theta_*(\tau) d\tau \end{aligned} \quad (1.11)$$

where

$$G(x, t; x_0, \tau) = \frac{1}{2\sqrt{\pi\alpha(t-\tau)}} e^{-\frac{(x-x_0)^2}{4\alpha(t-\tau)}}$$

is a function expressing the effect of an instantaneous point source for the equation of heat conductance (1.1). The solution of (1.1) as given by (1.11) will be unique if  $\theta_*(t)$  is specified for the boundary as a single-valued function of  $t$  and if the corresponding

integrals actually exist. It follows that  $\theta_2(t)$  is not necessarily conjugated with initial conditions.

Let us consider the integrals which appear on the right-hand side of (1.11). The integral

$$\int_0^t G(x, t; x_0(\tau), \tau) \cdot \psi(\tau; \theta_2) d\tau$$

is continuous throughout, including the line  $x_0(t)$ , if the function  $q$  is differentiable at all points, except perhaps the point  $t = 0$ , since in the neighborhood of this point it satisfies the inequality

$$|q| \leq \text{const } t^{-S} \quad \text{with } S \leq \frac{1}{2}.$$

The second integral in the right-hand portion of (1.11)

$$W(x, t) = \int_0^t \left[ G(x, t; x_0(\tau), \tau) v_0(\tau) - x \frac{\partial G(x, t; x_0(\tau), \tau)}{\partial x} \right] \theta_2(\tau) d\tau,$$

becomes discontinuous as it crosses the line  $x = x_0(t)$ . Here the following equation will hold (Ref. 13) for any continuous function  $\theta_2(t)$ :

$$W(x_0(t), t) - W(x_0(t), t) + \frac{1}{2} \theta_2(t).$$

We shall next consider the limit for  $x = x_0(t) = 0$  in (1.11). At the same time we have

$$\theta_2(t) = x \int_0^t G(x_0(t), t; x_0(\tau), \tau) \cdot \psi(\tau; \theta_2) d\tau + W(x_0(t), t) + \frac{1}{2} \theta_2(t).$$

If a solution of the problem exists, then function  $\theta_2(t)$  relating to this solution will satisfy equation

$$\begin{aligned} \theta_2(t) = & x \int_0^t G(x_0(t), t; x_0(\tau), \tau) \cdot \psi(\tau; \theta_2) d\tau + \\ & + 2 \int_0^t \left[ G(x_0(t), t; x_0(\tau), \tau) v_0(\tau) - x \frac{\partial G(x_0(t), t; x_0(\tau), \tau)}{\partial x} \right] \theta_2(\tau) d\tau, \end{aligned} \quad (1.12)$$

where

$$v_0(t) = \int_0^t \psi(\tau; \theta_2) d\tau. \quad (1.13)$$

Since  $G(x_0(t), t; x_0(\tau), \tau)$  and  $\frac{\partial G(x_0(t), t; x_0(\tau), \tau)}{\partial \tau}$  at  $\tau = t$  acquire a singularity of the form  $\frac{1}{\sqrt{t-\tau}}$ , it follows that when  $q(t; \theta_*)$  has the form

$\frac{g(t; \theta_*)}{\sqrt{t}}$ , where  $g(t; \theta_*)$  is everywhere a continuous function of its arguments, the system (1.12), (1.13) has a unique continuous solution. For a sufficiently small value of  $t$ , this solution can be found by the method of successive approximations. It is concluded directly that a solution of (1.1) exists and is unique, under conditions specified by (1.8)-(1.10).

The same thing can be demonstrated for a body of finite thickness. The value  $G$  in this case must be replaced by the Green's function corresponding to the boundary conditions on the underside of a plate.

3. We shall now consider an exact solution of the problem, for an elementary case. Let us solve a reverse problem, assuming that a constant temperature  $\theta_* = \text{const.}$  is established instantaneously on the body surface. This corresponds to a heat flow  $q$ , which will be determined. When the temperature of a surface is constant, the velocity of its translation in a space  $x_0(t)$  is likewise a known constant. Substituting the appropriate values of  $\theta_*$  and  $x_0(t)$  into (1.12), we obtain

$$\theta_* = \int_0^t [\theta_* \dot{x}_0(\tau) + 2\kappa\psi] \frac{1}{2\sqrt{\pi\kappa(t-\tau)}} e^{-\frac{x_0^2(t-\tau)}{4\kappa(t-\tau)}} d\tau. \quad (1.14)$$

From (1.14), by use of operational calculus, a function  $\psi(t; \theta_*)$  is derived, characterizing a heat flow penetrating the body:

$$\psi(t; \theta_*) = \theta_* \left\{ \frac{e^{-\frac{x_0^2(t)}{4\kappa t}}}{\sqrt{\pi\kappa t}} + \sqrt{\frac{t}{\pi}} \left( 1 + \frac{2}{\sqrt{\pi}} \int_0^{\frac{x_0(t)}{\sqrt{4\kappa t}}} e^{-x^2} dx \right) \right\}, \quad \sqrt{t} = \frac{x_0(t)}{2\sqrt{\kappa}} \quad (1.15)$$

The magnitude of the heat flow  $q$  can now be determined from (1.9).

In the case where no evaporation takes place,  $\dot{x}_0(t) = b = 0$ , and we obtain the familiar formula based on the theory of heat conductance

$$\psi = \frac{\theta_0}{\sqrt{\pi \kappa t}}, \text{ which in our case holds for sufficiently small values of } t.$$

When  $t \rightarrow \infty$ , a constant heat flow directed into the body becomes stabilized:

$$\psi(\infty) = -\frac{\dot{x}_0(t)}{\pi} \theta_0. \quad (1.16)$$

It will be noted that the same formula can be derived directly from the equation of heat conductance (1.1), if the solution is sought in the form of a heat wave propagating at a uniform speed

$$\theta(x, t) = \theta(y), \text{ where } y = -\frac{\dot{x}_0(t)}{\pi}(x - x_0(t)).$$

We then have

$$\theta = \theta_0 e^{\eta^2}.$$

Hence,

$$\frac{\partial \theta}{\partial x} = -\theta_0 \frac{\dot{x}_0(t)}{\pi} e^{\eta^2}.$$

For  $y = 0$  ( $x = x_0(t)$ ) this expression is identical with (1.16).

Thus, for the case where  $t \rightarrow \infty$ , the particular solution under consideration goes over asymptotically into a solution based on a heat wave propagating at uniform speed. As seen from (1.15), the rate at which this thermal process approaches a steady state is determined by the magnitude of parameter

$$\sqrt{\delta} = \frac{-\dot{x}_0(t)}{2\sqrt{\pi}}.$$

The velocity of the shifting evaporation front is low, as a rule, and the thermal diffusivity  $\kappa$  is of the order of unity (in the CGS system). Therefore the value of the parameter  $\sqrt{\delta}$  is small.

However, since the time required for the establishment of a quasi-stationary process such as a heat wave is of the order  $1/b$ , the time period involved may be quite large (by comparison with the characteristic time).

## II. Evaporation into an ambient space filled with gas at some distance from the body

1. Let a solid half-space, as in Section I, occupy initially a region  $x < 0$ , while the region  $x > 0$  is filled with a gas. It is further assumed that evaporation of the substance from the body surface, due to absorption of radiant energy, sets in, beginning at  $t = 0$ . The state of a gas mixture, it will be remembered, is determined principally by the processes of molecule transfer (such as viscosity, heat conductance and diffusion). In accordance with this basic concept, we shall neglect the derivatives of pressure in the Navier-Stokes and energy equations. This is a common assumption in the theory of combustion (both steady state and non-steady state processes) (Ref. 14). Unlike the processes considered in this work, combustion involves a chemical reaction. We shall further neglect both thermal and baric diffusion, as well as the heat flux and its dissipation resulting from diffusion. The gas parameters can now be determined from a system of differential equations

$$\begin{aligned} \rho \frac{\partial u}{\partial t} + \rho u \frac{\partial u}{\partial x} &= - \frac{\partial}{\partial x} \rho \frac{\partial u}{\partial x}, \\ \frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} &= 0, \\ \rho c_p \left( \frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} \right) &= \frac{\partial}{\partial x} \lambda \frac{\partial T}{\partial x}, \\ \rho \frac{\partial x}{\partial t} + \rho u \frac{\partial x}{\partial x} &= \frac{\partial}{\partial x} \rho D \frac{\partial x}{\partial x}. \end{aligned} \tag{2.1}$$



where  $c$  is the mass concentration of vapor. System (2.1) must be solved jointly with the equation for heat propagation in a solid body

$$\frac{\partial T_1}{\partial t} = \kappa \frac{\partial^2 T_1}{\partial x^2} \quad (2.2)$$

under the specific conditions of mass, concentration and energy conservation. The solution is derived for the resulting relationships on the evaporating surface:

$$\begin{aligned} x = x_0(t), \quad \dot{m} &= \rho_1 \dot{x}_0(t) = \rho_0 (u_0 - \dot{x}_0(t)) \approx \rho_0 u_0 \\ L \dot{m} &= -q - \lambda_1 \frac{\partial T_1}{\partial x} + \lambda \frac{\partial T}{\partial x} \\ \rho_0 (1 - c_0) (u_0 - \dot{x}_0(t)) - \rho_0 D \frac{\partial (1 - c)}{\partial x} &= 0 \end{aligned} \quad (2.3)$$

Subscript 0 refers to gas parameters near the surface; subscript 1, to values characterizing the solid body. Conditions on the evaporating surface  $x = x_0(t)$  are expressed by (2.3). To this system two more equations must be added, which describe a thermodynamic equilibrium between the solid body and its vapor:

$$x = x_0(t), \quad T_1 = T_0, \quad c_0 = c(T_0) \quad (2.4)$$

where  $c_0$  is the concentration of saturated vapor at a temperature  $T_0$ . At infinity we shall prescribe the following conditions:

$$\begin{aligned} x \rightarrow +\infty, \quad u &= 0, \quad c = 0, \quad p = p_{\infty}, \quad T = T_{\infty} \\ x \rightarrow -\infty, \quad T_1 &= T_{\infty} \end{aligned} \quad (2.5)$$

The initial conditions will be:

$$t = 0, \quad u = u(x, 0), \quad c = 0, \quad p = p(x, 0), \quad T = T(x, 0), \quad T_1 = T_1(x, 0) \quad (2.6)$$

In place of  $x$  a new independent variable - the Lagrange coordinate - is introduced:  $\eta = -u(t) + \int_{x_0}^x \rho dx$

## I. Simultaneous determination of temperatures and concentrations

$$\begin{aligned}
 \eta > -m(t) & \quad \frac{\partial T}{\partial t} = \beta \frac{\partial T}{\partial x^2}, \\
 & \quad \frac{\partial c}{\partial t} = \alpha \frac{\partial c}{\partial x^2}, \\
 \eta < -m(t) & \quad \frac{\partial T_1}{\partial t} = \beta_1 \frac{\partial T_1}{\partial x^2}, \quad \beta_1 = \kappa \rho_1^0, \\
 \eta = -m(t) & \quad T_1 = T_0, \quad c_0 = c_0(T_0), \\
 & \quad Lm(t) = -q - \lambda_1 \rho_1 \frac{\partial T_1}{\partial x} + \lambda_0 \frac{\partial T}{\partial x}, \\
 & \quad m(t) = -\frac{q}{1-c_0} \frac{\partial c}{\partial x}, \\
 \eta = \pm \infty, \quad T = T_\infty, \quad c = 0, \quad T_1 = T_\infty, \\
 t = 0, \quad T = T(x, 0), \quad c = 0, \quad T_1 = T_1(x, 0).
 \end{aligned}$$

## II. Determination of velocity and density

$$\begin{aligned}
 \frac{\partial u}{\partial t} &= \alpha \frac{\partial u}{\partial x^2}, \quad -\frac{1}{\rho^2} \frac{\partial \rho}{\partial t} = \frac{\partial u}{\partial x}, \\
 \eta &= -m(t), \quad \rho_0 \mu_0 = m(t), \\
 \eta &= \infty, \quad u = 0, \quad \rho = \rho_\infty, \\
 t = 0, \quad u = u(x, 0), \quad \rho = \rho(x, 0).
 \end{aligned}$$

The subsequent discussion will be concerned in the main with the fields of temperatures and concentrations.

2. To cite one example of an exact solution of the problem, let us assume that at the initial moment a half-space  $x > 0$  is filled with gas whose temperature is uniform throughout. The temperature of the solid body is likewise uniform everywhere:

$$T(x, 0) = T_\infty, \quad T_1(x, 0) = T_\infty$$

Starting from  $t = 0$  the solid body keeps evaporating, either because it is acted upon by a radiation flux equivalent to a heat flow of the form

$$q = \frac{q_0}{\sqrt{t}}.$$

( $Q \leq 0$  possibly depends on surface temperature), or because of mere interaction with the gas, since the body and the gas had different initial temperature.

From considerations of similarity it is concluded that the solution of problem I must depend on a single dimensionless variable  $z = \eta/\sqrt{at}$ :

$$T = T_{1\infty} + (T_{\infty} - T_{1\infty})\theta(z), \quad T_1 = T_{1\infty} + (T_{\infty} - T_{1\infty})\theta_1(z), \quad c = c(z).$$

The movement of the boundary is proportional to  $\sqrt{t}$ :

$$-m(t) = \eta_0(t) = z_0\sqrt{at} \quad (z_0 < 0).$$

Assuming  $z_0$ ,  $\theta_0$ ,  $c_0$  to be known, we shall find for functions  $\theta(z)$ ,  $\theta_1(z)$ ,  $c(z)$  expressions which will satisfy the original equations, the initial conditions and the conditions of infinity:

$$\begin{aligned} \theta(z) &= 1 - (1 - \theta_0) \frac{\operatorname{erfc} \frac{z}{2\sqrt{B/a}}}{\operatorname{erfc} \frac{z_0}{2\sqrt{B/a}}}, & \theta_1(z) &= \theta_0 \frac{2 - \operatorname{erfc} \frac{z}{2\sqrt{B/a}}}{2 - \operatorname{erfc} \frac{z_0}{2\sqrt{B/a}}}, \\ c(z) &= c_0 \frac{\operatorname{erfc} \frac{z}{2\sqrt{a/a}}}{\operatorname{erfc} \frac{z_0}{2\sqrt{a/a}}}. \end{aligned} \quad (2.7)$$

where

$$\operatorname{erfc} x = \frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-x^2} dx.$$

Since for  $\eta = -m(t)$ ,  $c_0 = c_0(T_0)$ , it follows that  $c_0 = c_0(\theta_0)$ . The remaining unknown values  $z_0$ ,  $\theta_0$  are determined from relationships expressing the laws of energy and component conservation. These, in our case, have the form

$$-kL(\theta_0)z_0 = -Q(\theta_0) - \theta_1'(z_0) + r\theta'(z_0), \quad (2.8)$$

$$-\frac{a}{2}z_0 = -\frac{c}{1-c_0}\theta'(z_0). \quad (2.9)$$

where

$$k = \frac{L_0}{2\lambda_1 \rho_1 (T_{\infty} - T_{100})}, \quad \bar{Q} = \frac{q}{\lambda_1 \rho_1 (T_{\infty} - T_{100})}, \quad r = \frac{\lambda_2}{\lambda_1}, \quad L(\theta) = \frac{L}{L_0}.$$

$L_0$  is the heat of evaporation at  $T = T_{100}$ .

We shall now transform the system (2.8), (2.9), introducing notations

$$w(z_0) = -r \frac{d}{dz_0} \ln \left( \operatorname{erfc} \frac{z_0}{2\sqrt{\beta_1/\alpha}} \right) = \frac{\exp \left( -\frac{z_0^2}{4\beta_1/\alpha} \right)}{\operatorname{erfc} \frac{z_0}{2\sqrt{\beta_1/\alpha}}} > 0,$$

$$w_1(z_0) = \frac{d}{dz_0} \ln \left( 2 - \operatorname{erfc} \frac{z_0}{2\sqrt{\beta_1/\alpha}} \right) = \frac{1}{\sqrt{\pi\beta_1/\alpha}} \frac{\exp \left( -\frac{z_0^2}{4\beta_1/\alpha} \right)}{\left( 2 - \operatorname{erfc} \frac{z_0}{2\sqrt{\beta_1/\alpha}} \right)}.$$

We then have

$$\theta'(z_0) = (1 - \theta_0) w(z_0), \quad \theta_1'(z_0) = \theta(z_0) w_1(z_0).$$

Considering that the specific heat of the body  $c_1$  and that of the vapor  $c_p$  are constant, the dependence of the heat of evaporation on temperature will be linear:

$$L(T_0) = L_0 + (c_p - c_1)(T_0 - T_{100}).$$

Consequently,

$$L(\theta) = 1 + \frac{c_p - c_1}{L_0} (T_{\infty} - T_{100}) \quad (2.10)$$

By use of (2.10), equation (2.8) can be rewritten in the form

$$\theta_0(z_0) = \frac{k_0 + w(z_0) - \bar{Q}}{w(z_0) + w_1(z_0) - k \frac{c_p - c_1}{L_0} (T_{\infty} - T_{100})} \quad (2.11)$$

If  $q$  is assumed independent of surface temperature ( $Q = \text{const.}$ ), the system of two transcendental equations (2.9), (2.11) will be reduced to a single equation which, on letting  $\eta = -\frac{z_0}{2\sqrt{\beta_1/\alpha}}$ , can be

be written as

$$y(1 + \operatorname{erf} y) = \frac{1}{\sqrt{\pi}} \frac{c_0}{1 - c_0} e^{-y^2},$$

since  $\operatorname{erfc}(-y) = 1 + \operatorname{erf} y$ . This equation has a solution, which is unique.

We can now solve problem II, which involves the distribution of velocity  $u(\eta, t)$  and of density. A solution is found in the form

$$u(\eta, t) = \frac{1}{\sqrt{at}} V(z), \quad p(\eta, t) = p(z).$$

Calculation yields

$$u(\eta, t) = \frac{c(z_0)}{\sqrt{at}} e^{\frac{z_0^2 - \eta^2}{4t}},$$

$$\frac{1}{p} = \frac{1}{p_\infty} + \frac{c(z_0)}{a} e^{\frac{z_0^2 - \eta^2}{4t}}.$$

where  $c(z_0)$  is determined by substituting  $u_0$  and  $p_0$  into the condition of mass conservation ( $\rho_0 u_0 = m$ ). We obtain

$$c(z_0) = -\frac{a}{p_\infty} \frac{z_0}{1 + z_0}.$$

The problem under consideration, it will be noted, remains self-similar also for the case of an arbitrary dependence of  $\mu, \lambda, D$  on temperature and concentration.

3. Problem I is a relatively complex nonlinear problem, hence an exact solution of it can apparently be found only for some particular cases. The above self-similar problem is the simplest of these cases. When the equations are of the same type as the equation for heat conductance, it is often possible to obtain another, quite simple yet rather interesting, solution, based on a wave propagating at uniform speed. We cannot however consider here the case of a wave

propagating in one direction only, since a solution of equations in this form will not satisfy all the boundary conditions involved. Inasmuch as deriving an exact solution under arbitrary initial and boundary conditions presents a highly complicated problem, we propose a method for arriving at an approximate solution. We shall transpose the boundary conditions for the case of  $\eta = \eta_0(t)$  to the line  $\eta = 0$ . Problem I, considerably simplified thanks to this assumption, can be formulated as shown below:

$$\begin{aligned} 0 < \eta < \infty, \quad \frac{\partial T}{\partial x} &= \beta \frac{\partial T}{\partial \eta}, \quad \frac{\partial c}{\partial x} = \sigma \frac{\partial c}{\partial \eta}, \\ -\infty < \eta < 0, \quad \frac{\partial T_1}{\partial x} &= \beta_1 \frac{\partial T_1}{\partial \eta}, \\ \eta = 0, \quad T_1 &= T_0, \quad c_0 = c(T_0), \\ -L \frac{\sigma}{1-c_0} \frac{\partial c}{\partial \eta} &= -q - \lambda_1 \rho_1 \frac{\partial T_1}{\partial \eta} + \lambda_p \frac{\partial T}{\partial \eta}, \\ t = 0, \quad T &= T(x, 0), \quad c = 0, \quad T_1 = T_1(x, 0). \end{aligned}$$

The condition  $\dot{m}(t) = -\frac{\sigma}{1-c_0} \frac{\partial c}{\partial \eta}$  serves to determine the law governing the shifting of the boundary, after the fields of temperature and concentrations have been defined. We are now ready to solve our problem. Let us denote by  $g_\beta(\eta, t; \xi, \tau)$  the Green function of the first boundary problem in the heat conductance equation, where  $\beta$  is the coefficient at the second derivative with respect to  $\eta$  for the half-line  $\eta > 0$ :

$$g_\beta(\eta, t; \xi, \tau) = \frac{1}{2\sqrt{\lambda\beta(t-\tau)}} \left\{ e^{-\frac{(\eta-\xi)^2}{4\beta(t-\tau)}} - e^{-\frac{(\eta+\xi)^2}{4\beta(t-\tau)}} \right\}.$$

Functions  $g_\beta(\eta, t; \xi, \tau)$ ,  $g_0(\eta, t; \xi, \tau)$  are obtained from  $g_\beta(\eta, t; \xi, \tau)$  by substituting  $\beta_1$  and  $\sigma$  for  $\beta$ . Assuming the surface temperature  $T(0, t) = T_0(t)$  to be known, we shall obtain for the functions

$T(\eta, t)$ ,  $c(\eta, t)$ ,  $T_1(\eta, t)$  expressions

$$T(\eta, t) = \int_0^\infty T(y, 0) g_p(\eta, t; y, 0) dy + \beta \int_0^t T_0(\tau) \frac{\partial g_p(\eta, t; 0, \tau)}{\partial y} d\tau, \quad (2.12)$$

$$c(\eta, t) = \sigma \int_0^t c_0 \frac{\partial g_0(\eta, t; 0, \tau)}{\partial y} d\tau, \quad (2.13)$$

$$T_1(\eta, t) = \int_{-\infty}^0 T_1(y, 0) g_p(\eta, t; y, 0) dy - \beta_1 \int_0^t T_0(\tau) \frac{\partial g_{p_1}(\eta, t; 0, \tau)}{\partial y} d\tau, \quad (2.14)$$

where  $c_0 = c(T_0)$ . The surface temperature  $T_0(t)$  is determined from the condition expressing the heat balance at the boundary. We shall write it in the form integrated with respect to time  $t$ :

$$-\sigma \int_0^t \frac{L}{1-c_0} \frac{\partial c}{\partial \eta} dt = \int_0^t \left[ -q - \lambda_1 \rho_1 \frac{\partial T_1}{\partial \eta} + \lambda_p \frac{\partial T}{\partial \eta} \right] dt. \quad (2.15)$$

The integral on the left-hand side of (2.15) can be rewritten in the form

$$\int_0^t \frac{L}{1-c_0} \frac{\partial c}{\partial \eta} dt = l(t) \int_0^t \frac{\partial c}{\partial \eta} dt \quad (2.16)$$

where, at the first approximation,

$$l(t) = \frac{L}{1-c_0}.$$

We shall now set up the partial derivatives of functions

$T(\eta, t)$ ,  $c(\eta, t)$ ,  $T_1(\eta, t)$  with respect to  $\eta$ . These functions are determined from eq. (2.12)-(2.14). We shall integrate them with respect to  $t$ , and shall obtain at the limit, for  $\eta \rightarrow 0$ :

$$\lim_{\eta \rightarrow 0} \int_0^t \frac{\partial T(\eta, \lambda)}{\partial \eta} d\lambda = \int_0^\infty \int_0^t \frac{\partial g_p(0, \lambda; y, 0)}{\partial \eta} dy d\lambda - \int_0^t T_0(\tau) \frac{\partial}{\partial \eta} \frac{d\tau}{\sqrt{\pi \beta_1 (t-\tau)}}. \quad (2.17)$$

$$\lim_{\eta \rightarrow 0} \int_0^t \frac{\partial c(\eta, \lambda)}{\partial \eta} d\lambda = - \int_0^t c_0 \frac{d\tau}{\sqrt{\pi \sigma (t-\tau)}}. \quad (2.18)$$

$$\begin{aligned} \lim_{\eta \rightarrow 0} \int_0^t \frac{\partial T_1(\eta, \lambda)}{\partial \eta} d\lambda = & \int_0^\infty \int_0^t T_1(y, 0) \frac{\partial g_{p_1}(0, \lambda; y, 0)}{\partial \eta} dy d\lambda + \\ & + \int_0^t T_0(\tau) \frac{d\tau}{\sqrt{\pi \beta_1 (t-\tau)}}. \end{aligned} \quad (2.19)$$

Substituting (2.17)-(2.19) into (2.15), we obtain by use of (2.16) an Abel equation expressing the function

$$\Phi(T_0) = \left( \frac{\lambda_1 \rho_1}{V \lambda_1} + \frac{\lambda_2}{V \lambda_2} \right) T_0(t) + \sqrt{\gamma} l(t) c_0(T_0)$$

and thus arrive at the following equation for determining surface temperature

$$\Phi(T_0) = - \int_0^t \frac{F d\tau}{V \sqrt{\pi(t-\tau)}} \quad (2.20)$$

where

$$F = q(t; T_0) + \lambda_1 \rho_1 \int_0^t T_1(y, 0) \frac{\partial g_{\rho_1}(0, t; y, 0)}{\partial \eta} dy - \lambda_2 \int_0^t T(y, 0) \frac{\partial g_{\rho}(0, t; y, 0)}{\partial \eta} dy.$$

If the value of the heat flux does not depend on surface temperature, then  $F$  is a function of time alone, and formula (2.20) gives an implicit expression for surface temperature.

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#### REFERENCES

1. B.A. Grigoryev. Izvestiya AN SSSR, Otdel tekhnicheskikh nauk (Bulletin of the Academy of Sciences USSR, Section of Technical Sciences) No. 1, 1958.
2. V.I. Danilovskaya, ibidem, Mekhanika i mashinostroyeniye (Mechanics and Machine Building) No. 3, 1959.
3. L.I. Rubinstein. Doklady AN SSSR (Academy of Sciences USSR. Reports) 58, No. 2, 1947.



References cont'd

4. V.G. Melamed. Vestnik Moskovskogo Gosudarstvennogo Universiteta (Bulletin of Moscow State University) No. 1, 1959.
5. D.A. Frank-Kamenetsky. Diffusion and Heat Transfer in Chemical Kinetics. Publ. Academy of Sciences USSR, 1947.
6. V.G. Levich. Physicochemical Hydrodynamics. Fizmatgiz (State Publishing House for the Literature in Physics and Mathematics), Moscow, 1959.
7. G.G. Chernyi. Izvestiya AN SSSR, Otdel tekhnicheskikh nauk (Bulletin, Academy of Sciences USSR, Section of Technical Sciences) No. 12, 1954.
8. V.A. Kireye. A Course in Physical Chemistry. Goskhimizdat (State Publishing House for Chemical Literature), Moscow, 1956.
9. L.D. Landau and E.M. Lifshits. Mechanics of Continuous Media. Gostekhteorizdat (State Publishing House for Theoretical and Technical Literature), 1954.
10. R.Ya. Kucherov, L.E. Rikenglaz: Zhurnal eksperimental'noy i teoreticheskoy fiziki (J. of Exp. and Theor. Physics) 37, No. 1, 1959.
11. M. Baranayev. Zhurnal fizicheskoy khimii (J. of Physical Chem.) 13, 1959, p. 1635. /Tr. note: This reference is erroneous; I could not find it in the entire year 1959/
12. A.N. Tikhonov and A.A. Samarsky. Equations of Mathematical Physics, Gostekhteorizdat, Moscow, 1953.
13. V.I. Smirnov. A Course in Higher Mathematics, 4, Gostekhteorizdat, 1957.
14. G. Rosen. J. Chem. Physics. 30, No. 1, 1959.

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